

Local thermodynamical equilibrium and the β frame for a quantum relativistic fluid

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We discuss the concept of local thermodynamical equilibrium in relativistic hydrodynamics in a quantum statistical framework without an underlying kinetic description suitable for strongly interacting fluids. We show that the appropriate definition of local equilibrium naturally leads to the introduction of a relativistic hydrodynamical frame in which the four-velocity vector is the one of a relativistic thermometer at equilibrium with the fluid, parallel to the inverse temperature four-vector β , which then becomes a primary quantity. We show that this frame is the most appropriate for the expansion of stress-energy tensor from local thermodynamical equilibrium and that therein the local laws of thermodynamics take on their simplest form. We discuss the difference between the β frame and Landau frame and present an instance where they differ.

I. INTRODUCTION

In recent years, relativistic hydrodynamics has drawn much attention. Part of the revived interest [1, 2] is owing to the successful hydrodynamic description of the Quark Gluon Plasma formed in collisions of nuclei at very high energy [3–10]. It is also known that hydrodynamics can be applied to a large portion of the phase diagram of condensed matter systems presenting quantum critical points [11–14]. Focusing on the Quark Gluon Plasma, close to the QCD critical temperature, the system is made of strongly interacting quantum fields and does not apparently allow a description in terms of weakly-interacting quasiparticles [15, 16]. Thus, the use of kinetic theory to describe it can be questioned, and yet, because the microscopic interaction length is small compared to its overall size, the system is actually a fluid.

In principle, hydrodynamics does not need an underlying kinetic theory nor a discrete particle substratum, even if its use can be very effective to obtain useful relations [17]. Hydrodynamic is, in essence, the continuity equation of the *mean* value of the stress-energy tensor (and charge current) operator, which, being primarily expressed in terms of quantum fields, does not need a single-particle distribution function $f(x, p)$. In fact, its momentum-space integral expression in terms of $f(x, p)$ can be obtained under special conditions, those which make kinetic approach suitable [18].

Consequently, all basic concepts in hydrodynamics, such as the flow velocity, should be defined independently of kinetic theory and of the single-particle distribution function. This also applies to one of the most used and basic notions of hydrodynamics, that is local thermodynamic equilibrium (LTE). In many textbooks, LTE is defined within a kinetic framework by making the collisional integral of the (relativistic) Boltzmann equation vanishing [18]. However, this is not the most general definition; in quantum statistical mechanics LTE can be defined as a maximum of the entropy with specific constraints [19]. In this work, we will show that, in the relativistic context, such a definition naturally leads to the introduction of a four-vector field which functions as a hydrodynamical velocity, that is a hydrodynamical frame: the inverse temperature four-vector β .

Thus far, this four-vector field has been mostly considered as a secondary quantity, formed by multiplying the invariant temperature $1/T$ by an otherwise defined velocity four-vector u . Recently, Van and Biro [20] have argued that β in fact defines a new independent frame, a conclusion that we fully support. Indeed, in this paper, we will reinforce it and demonstrate it in the most general quantum relativistic framework without resorting to kinetic arguments. We will show that it is much more natural and convenient to take β as a primordial field related to

the concept of LTE, so that the four-velocity of a relativistic fluid can be defined starting from the β field and not vice-versa:

$$u(x) \equiv \frac{\beta}{\sqrt{\beta^2}}$$

The paper is organized as follows. In sect. II, III we review the concept of local thermodynamical equilibrium in relativistic quantum statistical mechanics and introduce the β frame. In sect. IV we will show how to operationally define the β vector through an ideal relativistic thermometer, providing a better insight of its physical meaning. In sect. V we will discuss the form of the stress-energy tensor in the β frame, in sect. VI we will point out the difference between β and Landau frames; finally in sect. VII we will discuss the separation between the ideal and dissipative part of the stress-energy tensor.

Notation

In this paper we use the natural units, with $\hbar = c = K = 1$. The Minkowskian metric tensor is $\text{diag}(1, -1, -1, -1)$; for the Levi-Civita symbol we use the convention $\epsilon^{0123} = 1$. We will use the relativistic notation with repeated indices assumed to be saturated, however contractions of indices will be sometimes denoted with dots, e.g. $u \cdot T \cdot u \equiv u_\mu T^{\mu\nu} u_\nu$. Operators in Hilbert space will be denoted by a large upper hat, e.g. \hat{R} while unit vectors with a small upper hat, e.g. \hat{v} . We will work with a symmetric stress-energy tensor with an associated vanishing spin tensor.

II. LOCAL THERMODYNAMICAL EQUILIBRIUM IN RELATIVISTIC QUANTUM STATISTICAL MECHANICS

In the most general framework of quantum statistical mechanics, LTE is defined by the maximization of the Von Neumann entropy $S = -\text{tr}(\hat{\rho} \log \hat{\rho})$, $\hat{\rho}$ being the density operator, with the constraints of fixed densities of energy, momentum and charge [19]. As has been mentioned in the Introduction, such a definition does not require any underlying kinetic theory; the only requirement is that densities significantly vary over distances much larger than the typical microscopic scale.

In non-relativistic thermodynamics, the LTE definition is an unambiguous one and leads to a unique density operator obtained by maximizing the function, at any given time t

$$-\text{tr}(\hat{\rho} \log \hat{\rho}) + \int d^3x \, b(\mathbf{x}, t) [\langle \hat{h}(\mathbf{x}, t) \rangle - h(\mathbf{x}, t)] - b(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot [\langle \hat{\boldsymbol{\pi}}(\mathbf{x}, t) \rangle - \boldsymbol{\pi}(\mathbf{x}, t)] - \xi(\mathbf{x}, t) [\langle \hat{q}(\mathbf{x}, t) \rangle - q(\mathbf{x}, t)], \quad (1)$$

where h , $\boldsymbol{\pi}$ and q are the actual values of the energy, momentum and particle (or charge) density respectively; $b = 1/T$ and $\xi = \mu/T$ are point-dependent Lagrange multiplier, as well as \mathbf{v} whose meaning is the mean velocity of the particles. The symbol $\langle \rangle$ stands for the renormalized mean value of the operators:

$$\langle \hat{A} \rangle = \text{tr}(\hat{\rho} \hat{A})_{\text{ren}}.$$

For the simple case of a free quantum field theory this corresponds to the normal ordering of creation and destruction operator and if \hat{A} is quadratic in the fields to the subtraction of its vacuum expectation value:

$$\text{tr}(\hat{\rho} \hat{A})_{\text{ren}} = \text{tr}(\hat{\rho} : \hat{A} :) = \text{tr}(\hat{\rho} \hat{A}) - \langle 0 | \hat{A} | 0 \rangle.$$

The density operator $\hat{\rho}_{\text{LE}}$ resulting from the maximization of (1) is called LTE density operator:

$$\hat{\rho}_{\text{LE}} = \frac{1}{Z_{\text{LE}}} \exp \left[- \int d^3x \, b(\mathbf{x}, t) \hat{h}(\mathbf{x}, t) + b(\mathbf{x}, t) \mathbf{v}(\mathbf{x}, t) \cdot \hat{\boldsymbol{\pi}}(\mathbf{x}, t) - \xi(\mathbf{x}, t) \hat{q}(\mathbf{x}, t) \right], \quad (2)$$

where Z_{LE} is the normalizing factor making $\text{tr} \hat{\rho}_{\text{LE}} = 1$. The values of the Lagrange multipliers b , \mathbf{v} and ξ are obtained enforcing $\langle \hat{A} \rangle = A$, where A is respectively the actual value of the energy, momentum and charge of the system. A galileian transformation does not change the resulting density operator except for a shift of the parameter \mathbf{v} , but the entropy $S = -\text{tr}(\hat{\rho}_{\text{LE}} \log \hat{\rho}_{\text{LE}})$ is invariant.

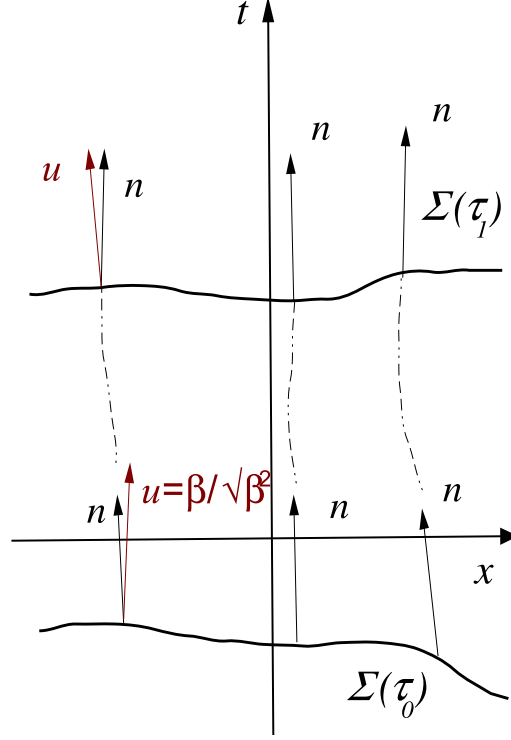


FIG. 1. Spacelike hypersurfaces $\Sigma(\tau)$ and their normal vector n defining local thermodynamical equilibrium for a relativistic fluid in Minkowski spacetime. If the β field has vanishing vorticity, β can be chosen parallel to n at each point; conversely, the normal vector and the β direction do not coincide (see sect. III).

Extending the definition of LTE to quantum relativistic statistical mechanics is not straightforward because energy density and momentum density are frame-dependent quantities in a much stronger fashion than in non-relativistic mechanics. To make it fully covariant, it is necessary to fix a τ -parametric family of spacelike hypersurfaces $\Sigma(\tau)$. The timelike unit vector field $n(x)$ normal to the surfaces defines world lines of observers (see fig. 1), yet the parameter τ , in general, does not coincide with the proper time of comoving clocks. As it is known, for orthogonal surfaces to exist, the field $n(x)$ must be vorticity-free, i.e. it ought to fulfill the equation:

$$\epsilon_{\mu\nu\rho\sigma} n^\nu (\partial^\rho n^\sigma - \partial^\sigma n^\rho) = 0 \quad (3)$$

For the frame having n as time direction, we enforce the mean energy-momentum and charge density to be the actual ones everywhere:

$$\begin{aligned} n_\mu \text{tr}(\hat{\rho}_{\text{LE}} \hat{T}^{\mu\nu}(x))_{\text{ren}} &= n_\mu \langle \hat{T}^{\mu\nu}(x) \rangle_{\text{LE}} \equiv n_\mu T_{\text{LE}}^{\mu\nu}(x) = n_\mu T^{\mu\nu}(x) \\ n_\mu \text{tr}(\hat{\rho}_{\text{LE}} \hat{j}^\mu(x))_{\text{ren}} &= n_\mu \langle \hat{j}^\mu(x) \rangle_{\text{LE}} \equiv n_\mu j_{\text{LE}}^\mu(x) = n_\mu j^\mu(x) \end{aligned} \quad (4)$$

where \hat{T} is the stress-energy tensor operator and \hat{j} the conserved current (if any). The function to be maximized as a function of $\hat{\rho}$, at any τ , reads:

$$-\text{tr}(\hat{\rho} \log \hat{\rho}) + \int_{\Sigma(\tau)} d\Sigma n_\mu \left[\left(\langle \hat{T}^{\mu\nu}(x) \rangle - T^{\mu\nu}(x) \right) \beta_\nu(x) - \left(\langle \hat{j}^\mu(x) \rangle - j^\mu(x) \right) \xi(x) \right]$$

where $d\Sigma$ is the measure (in the Minkowski spacetime) of the hypersurface, β is, by definition, the inverse temperature four-vector and ξ a scalar field of Lagrange multipliers whose meaning will be clear shortly. The solution is:

$$\hat{\rho}_{\text{LE}} = \frac{1}{Z_{\text{LE}}} \exp \left[- \int_{\Sigma(\tau)} d\Sigma n_\mu \left(\hat{T}^{\mu\nu}(x) \beta_\nu(x) - \xi(x) \hat{j}^\mu(x) \right) \right] \quad (5)$$

This covariant form of an equilibrium density operator, to our knowledge, was first written down by Weldon [22]. It is clear that the operator in (5) does depend on the particular hypersurface $\Sigma(\tau)$ (whence on the field n). Accordingly, the mean values T_{LE} and j_{LE} depend on the hypersurface Σ , hence, in general one can write

$$T_{\text{LE}}^{\mu\nu} = T_{\text{LE}}^{\mu\nu}[\beta, \xi, n] \quad j_{\text{LE}}^\mu = j_{\text{LE}}^\mu[\beta, \xi, n],$$

so that even the β field, obtained as a solution of the eq. (4), will depend on n :

$$\beta_\mu T_{\text{LE}}^{\mu\nu}[\beta, \xi, n] = \beta_\mu T^{\mu\nu} \quad \beta_\mu j_{\text{LE}}^\mu[\beta, \xi, n] = \beta_\mu j^\mu, \quad (6)$$

where the square brackets mean that the dependence of the currents on the fields β, ξ and n is in general functional (e.g. there could be a dependence on the derivatives). For all means to be independent of it, the divergence of the integrand should vanish - provided that some boundary conditions are enforced [23] - a condition which is met if:

$$\partial_\mu \beta_\nu + \partial_\nu \beta_\mu = 0 \quad \partial_\mu \xi = 0. \quad (7)$$

The above equations just define the known condition of global thermodynamical equilibrium (GTE) for a relativistic fluid. The first equation states that β is a timelike Killing vector field and this is enough to ensure stationarity of the operator and of the mean values of time-dependent operators constructed with the density operator (5).

Going back to local equilibrium, as long as the field $n(x)$ is not specified, the definition of LTE is ambiguous¹. To show that there is a preferential choice thereof, one can calculate the total entropy by using (2)

$$S = -\text{tr}(\hat{\rho}_{\text{LE}} \log \hat{\rho}_{\text{LE}})_{\text{ren}} = \log Z_{\text{LE}} + \int_{\Sigma(\tau)} d\Sigma n_\mu (T_{\text{LE}}^{\mu\nu} \beta_\nu - \xi j_{\text{LE}}^\mu). \quad (8)$$

A crucial and mostly unspoken assumption in relativistic extension of thermodynamics is that $\log Z_{\text{LE}}$ can be written as an integral over the hypersurface Σ of a four-vector field, defined as *thermodynamical potential current* ϕ^μ , depending on the functions β and ξ

$$\log Z_{\text{LE}}(\tau) = \log \text{tr} \left(\exp \left[- \int_{\Sigma(\tau)} d\Sigma n_\mu (\hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu) \right] \right) = \int_{\Sigma(\tau)} d\Sigma n_\mu \phi^\mu[\beta, \xi, n]. \quad (9)$$

This assumption is necessary for the existence of an entropy current s^μ which is one of the starting points of Israel's formulation of relativistic hydrodynamics [24]. Although (9) should be proved, we account it in this work as an *ansatz*. Hence, in view of (8) and (9), the entropy current reads:

$$s^\mu = \phi^\mu + T_{\text{LE}}^{\mu\nu} \beta_\nu - \xi j_{\text{LE}}^\mu + s_T^\mu(n) \quad (10)$$

where $s_T(n)$ is an arbitrary four-vector field orthogonal to n . Note from (9) that also ϕ is defined up to an arbitrary four-vector field orthogonal to n . It should be emphasized that in nonequilibrium situations, since $\partial_\mu s^\mu \neq 0$, the total entropy S in (8) is a frame-dependent quantity, as it varies with the integration hypersurface Σ . Indeed, like T_{LE} and j_{LE} , the local currents ϕ will also depend on the hypersurface Σ (see its dependence on n in eq. 9).

A great simplification would be achieved if $n = \hat{\beta} = \beta_\mu / \sqrt{\beta^2}$, as the number of independent variables on which mean values depend would be reduced. With this choice the eq. (4) would become

$$\beta_\mu T_{\text{LE}}^{\mu\nu}[\beta, \xi] = \beta_\mu T^{\mu\nu} \quad \beta_\mu j_{\text{LE}}^\mu[\beta, \xi] = \beta_\mu j^\mu, \quad (11)$$

where the right hand sides contain the true values at each point. This choice is what we define as *β frame*.

Indeed, setting $n = \hat{\beta}$ is possible only if the β field solution of the eq. (6), also fulfills the equations (3). In fact, this equation does not apply even for the simple case of a rigid velocity field, which is actually a global thermodynamical equilibrium one (see Appendix A). Notwithstanding, also in the vortical case, it is possible to find a proper definition of the n field based on β , as it will be shown in the next section.

From a physical viewpoint, the β frame is identified by the four-velocity of a relativistic thermometer at local equilibrium with the system, what will be discussed in detail in sect. IV. This frame has more peculiar features. As an example, let us contract the equation (10) with n_μ , which enables us to use eq. (4) to replace the local equilibrium averages T and j with their actual values

$$s^\mu n_\mu = n_\mu \phi^\mu + n_\mu T^{\mu\nu} \beta_\nu - \xi n_\mu j^\mu. \quad (12)$$

¹ Note that the field n does not necessarily coincide with the hydrodynamic velocity field u albeit, as we will see, it is related to it.

The left hand side is the entropy density seen by the observer moving with four-velocity n . If $n_\mu = \hat{\beta}_\mu$, the eq. (12) is manifestly the basic relation of thermodynamics expressing the proper entropy density s as a function of proper energy and charge density

$$\sqrt{\beta^2} s = \beta \cdot \phi + \beta_\mu \beta_\nu T^{\mu\nu} - \xi \beta_\mu j^\mu, \quad (13)$$

provided that $\beta^2 = 1/T^2$ and $\xi = \mu/T$, what makes the physical meaning of β and ξ apparent. Indeed, introducing the symbols ρ for the proper energy density and n for the proper charge density

$$Ts = T^2 \beta \cdot \phi + \rho - \mu n, \quad (14)$$

where

$$\rho = \frac{\beta_\mu \beta_\nu T^{\mu\nu}}{\beta^2} \quad n = \frac{\beta_\mu j^\mu}{\sqrt{\beta^2}}. \quad (15)$$

The equation (14) tells us that the β frame is the one where the basic thermodynamic relation between proper entropy density and proper (true) energy and charge densities takes on its simplest form. In different frames, this relation is to be obtained contracting with a vector different from n and it may thus contain additional terms (most likely of the second order in derivatives). We conclude this section by noting that in the familiar global thermodynamical equilibrium, it is known [23, 24] that the four-vector field $\phi^\mu = p\beta^\mu$ where p is the pressure, hence the (14) can be written in the more familiar form

$$Ts = p + \rho - \mu n. \quad (16)$$

In fact, at local thermodynamical equilibrium, the thermodynamic potential current ϕ may have additional terms depending, e.g., on derivatives of the β and ξ fields. If these additional terms do have a longitudinal (along β) component, then the above equation is to be replaced by the most general (14).

III. LOCAL THERMODYNAMICAL EQUILIBRIUM FOR A GENERAL β FIELD

For a general, non vorticity-free field β , the identification $n = \hat{\beta}$ is not possible and must be modified. One can iteratively construct a field $n(x)$ which fulfills eq. (3) and, at the same time, reproducing the known features of global thermodynamical equilibrium with rotation (see discussion in [23, 25]). Take:

$$b_\mu^{(1)} = \beta_\mu + \frac{1}{2} x^\nu (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) \quad (17)$$

Clearly, $\partial_\mu b_\nu^{(1)} - \partial_\nu b_\mu^{(1)} \approx O(\partial^2)$. Iteratively, one can add to $b_\mu^{(1)}$ higher order derivative terms which are antisymmetric in $\mu\nu$ to eliminate gradients at some order. For instance,

$$b_\mu^{(2)} = \beta_\mu + x^\nu \left(1 + \frac{x \cdot \partial}{3} \right) \left[\frac{1}{2} (\partial_\mu \beta_\nu - \partial_\nu \beta_\mu) \right] \quad (18)$$

implies $\partial_\mu b_\nu^{(2)} - \partial_\nu b_\mu^{(2)} \approx O(\partial^3)$. Thereby, we can construct a field $b(x)$ with vanishing external derivative

$$\partial_\mu b_\nu - \partial_\nu b_\mu = 0, \quad (19)$$

which can be used to define the LTE hypersurfaces $\Sigma(\tau)$, i.e:

$$n(x) \equiv \hat{b}(x).$$

because, as it is apparent, if b fulfills eq. (3), any field collinear to it will. Hence, from eq. (17)-(18) we generalize to all orders, defining ϖ :

$$\beta_\nu(x) \equiv b_\nu(x) + \varpi_{\nu\lambda}(x) x^\lambda, \quad (20)$$

with:

$$\varpi_{\nu\lambda}(x) = -\frac{1}{2} (\partial_\nu \beta_\lambda - \partial_\lambda \beta_\nu) - \frac{1}{6} (x^\rho \partial_\rho \partial_\nu \beta_\lambda - x^\rho \partial_\rho \partial_\lambda \beta_\nu) + \dots \quad (21)$$

It should be pointed out that the obtained expressions (20) is in full agreement with the global equilibrium case ($\partial_\rho \varpi_{\mu\nu} = 0$), where the tensor ϖ is constant and the higher derivatives of the β field vanish [23]; b is constant as well and the eq. (20) becomes the general Killing vector field in Minkowski spacetime.

With the same LTE density operator as in eq. (5), the field β is now the solution of a modified version of the (11) enforcing the equality of the mean-energy and momentum density

$$\begin{aligned} b_\mu T_{\text{LE}}^{\mu\nu}[\beta, \xi] = b_\mu T^{\mu\nu} &\implies (\beta_\mu - \varpi_{\mu\lambda}[\beta]x^\lambda)T_{\text{LE}}^{\mu\nu}[\beta, \xi] = (\beta_\mu - \varpi_{\mu\lambda}[\beta]x^\lambda)T^{\mu\nu} \\ b_\mu j_{\text{LE}}^\mu[\beta, \xi] = b_\mu j^\mu &\implies (\beta_\mu - \varpi_{\mu\lambda}[\beta]x^\lambda)j_{\text{LE}}^\mu[\beta, \xi] = (\beta_\mu - \varpi_{\mu\lambda}[\beta]x^\lambda)j^\mu, \end{aligned} \quad (22)$$

with ϖ given by (21). We stress that, for a vorticious β field, it is not possible to restore eq. (11) instead of (22) to determine β , for T_{LE} requires a LTE density operator (5) to be defined and this in turn *demands* the constraints in the specific form (4) with n vorticity-free.

By using (5) and (9), one can find an expression of entropy current

$$s^\mu = \phi^\mu + T_{\text{LE}}^{\mu\nu}\beta_\nu - \xi j_{\text{LE}}^\mu + s_T^\mu(n). \quad (23)$$

The entropy density in the local rest frame of the fluid is obtained by contracting (23) with β

$$\sqrt{\beta^2}s = \phi^\mu\beta_\mu + \beta_\mu T_{\text{LE}}^{\mu\nu}\beta_\nu - \xi\beta_\mu j_{\text{LE}}^\mu + \beta \cdot s_T(n).$$

However, unlike in the non-vorticious case, replacing the local equilibrium mean values with the true ones is not straightforward.

IV. TEMPERATURE AND THERMOMETERS IN RELATIVITY

As has been mentioned in the Introduction, one can introduce a four-velocity u by means of the equation:

$$u(x) \equiv \frac{\beta}{\sqrt{\beta^2}}$$

once β is defined in terms of LTE. This definition, quite formal, can in fact be complemented by a more physical one introducing the notion of an ideal relativistic thermometer. Just as in classical thermodynamics, this is, by definition, a "small" object able of instantaneously achieve thermodynamical equilibrium with the system in contact with it. Besides, it should have some macroscopic internal property (such as size, resistivity etc.) which varies as a function of temperature, so that it can be used to define a scale thereof.

In the relativistic context, an ideal thermometer can exchange both energy and momentum with the system, and therefore its response is not limited to a change of its internal property gauging the temperature but it also includes a change of its four-velocity. In other words, once in contact with the system, the idealized relativistic thermometer will move at some finite speed which is determined by the local equilibrium conditions. Now, the discussion gets easier considering both the system and the thermometer small yet finite. If the thermometer attains full thermodynamical equilibrium with the system, the entropy will be maximal with respect to energy and momentum exchange, thus we can write (the subscript T refers to the thermometer quantities)

$$\frac{\partial S}{\partial P^\mu} = \frac{\partial S_T}{\partial P_T^\mu}, \quad (24)$$

keeping the proper volumes and the conserved charges fixed. Now, let us suppose that the the system is so small that β and ξ are essentially constant over the system and thermometer volumes so as to taking them out of the integral sign in eq. (8) ²

$$S = \log Z_{\text{LE}} + \int d\Sigma u_\mu (T^{\mu\nu}\beta_\nu - \xi j^\mu) \simeq \log Z_{\text{LE}} + \beta_\nu \int d\Sigma_\mu T^{\mu\nu} - \xi \int d\Sigma_\mu j^\mu = \log Z_{\text{LE}} + \beta_\nu P^\nu - \xi Q$$

where we have used the (4). Note that P and Q do not depend on the frame u because the divergences of T and j are assumed to vanish (interaction energy between system and thermometer is negligible by assumption). Hence,

² Henceforth, we will use the shorthand $d\Sigma_\mu$ for $d\Sigma n_\mu$

according to eq. (24) and keeping in mind the basic relations of equilibrium relativistic thermodynamics which express the mean values of energy-momentum as derivatives of $\log Z_{\text{LE}}$ we obtain

$$\beta_\nu = \beta_{\nu\text{T}}.$$

The above equation implies that a relativistic thermometer in thermodynamical equilibrium with the system will mark the temperature $T_0 = 1/\sqrt{\beta^2}$ and move with a speed $\beta/\sqrt{\beta^2}$. In this case, the thermometer is defined as *comoving* and the marked temperature is generally referred to as the local temperature.

Alternatively, one can retain a more traditional definition of an ideal thermometer as a “small” object endowed with a temperature gauge and able of instantaneously achieve thermodynamical equilibrium with the system in contact with it *with respect to energy exchange*; its velocity v can be externally imposed. According to the generally accepted extension of thermodynamics to relativity, one has to choose the frame where the thermometer is at rest and therein enforce the condition of maximal entropy with respect to only *energy* exchange

$$\frac{\partial S}{\partial E} = \frac{\partial S_{\text{T}}}{\partial E_{\text{T}}}, \quad (25)$$

which results in the equality of the time components of the β vectors in that frame:

$$\beta^0 = \beta_{\text{T}}^0$$

or

$$\beta \cdot v = \frac{1}{T_{\text{T}}}.$$

In conclusion, a thermometer moving with four-velocity v in a system in local thermodynamical equilibrium, characterized by a four-vector field β , will mark a temperature which is equal to

$$T_T = \frac{1}{\beta(x) \cdot v}. \quad (26)$$

As the scalar product of two timelike unit vectors $u \cdot v \geq 1$ and

$$u \cdot v = 1 \quad \text{iff } u = v$$

one has, according to (26)

$$T_T \leq T_0 = \frac{1}{\sqrt{\beta^2}} \quad T_T = T_0 \quad \text{iff } u = v,$$

that is the temperature marked by an idealized thermometer is maximal if it moves with the same four-velocity of the fluid. Thereby, we can establish a thought operational procedure to define a four-velocity of the fluid based on the notion of LTE at the spacetime point x :

- put (infinitely many) ideal thermometers in contact with the relativistic system at the spacetime point x , each with a different four-velocity v ;
- the ideal thermometer marking the *highest* value T_0 moves, by definition, with the four-velocity $u(x) = T_0 \beta(x) = 1/\sqrt{\beta^2}$.

V. THE STRESS-ENERGY TENSOR IN THE β FRAME

Relativistic hydrodynamics is, basically, the conservation of the mean values of energy and momentum as expressed by the continuity equation

$$\partial_\mu T^{\mu\nu} = \partial_\mu \text{tr}(\hat{\rho} \hat{T}^{\mu\nu})_{\text{ren}} = \text{tr}(\hat{\rho} \partial_\mu \hat{T}^{\mu\nu})_{\text{ren}} = 0, \quad (27)$$

where $\hat{\rho}$ is the actual, time-independent, density operator in the Heisenberg representation. In the above equation $T^{\mu\nu}$ is thus the true mean value of the stress-energy tensor and the (27) makes it clear that the conservation of the mean value stems from the more fundamental conservation equation of the corresponding quantum operator.

If, at some initial "time" τ_0 , the system is known to be at local thermodynamical equilibrium, one can take the actual, time-independent, density operator as the one in eq. (5) provided that both the spacelike hypersurface Σ and the operators \hat{T} , \hat{j} are evaluated at τ_0 . Consider now the evolution in τ of the local thermodynamical equilibrium hypersurface Σ ; one can then rewrite the actual density operator in terms of the operators at the present time τ by means of the Gauss' theorem:

$$-\int_{\Sigma(\tau_0)} d\Sigma n_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \hat{j}^\mu \xi \right) = -\int_{\Sigma(\tau)} d\Sigma n_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \hat{j}^\mu \xi \right) + \int_{\Omega} d^4x \left(\hat{T}^{\mu\nu} \partial_\mu \beta_\nu - \hat{j}^\mu \partial_\mu \xi \right), \quad (28)$$

where Ω is the spacetime region enclosed by the two hypersurface $\Sigma(\tau_0)$ and $\Sigma(\tau)$ and the timelike hypersurface at their boundaries (see e.g. fig. 1). The above formula requires that the flux of $\hat{T}^{\mu\nu} \beta_\nu(x) - \hat{j}^\mu \xi(x)$ vanish at those boundaries. The first term on the right hand side of eq. (28) is just the local thermodynamical equilibrium exponent at "time" τ . If the evolution of the stress energy tensor and current operators are such that the system keeps close to a situation of local thermodynamical equilibrium, the second term on the right hand side can be considered as a perturbation with respect to the first term and, accordingly, an expansion can be made in the gradients of the β and ξ fields. This decomposition has been put forward by [26], with a supplementary factor $\exp[\varepsilon(\tau - \tau')]$ in the integrand of the second term on the right hand side of eq. (28) and subsequent limit $\varepsilon \rightarrow 0$; with this prescription the "gradient" perturbation is turned on adiabatically from the far past $\tau_0 = -\infty$ and one finds the correct Green-Kubo formulae of transport coefficients [27].

One can recognize in the first term of the exponent of the right hand side of (28) the local thermodynamical equilibrium operator at the hypersurface $\Sigma(\tau')$. As has been mentioned, if the gradients of the β and ξ fields are small enough, one can write an expansion starting from the local thermodynamical equilibrium term through an iterated use of the Kubo identity, generating an expansion of the stress-energy tensor in the gradients

$$T^{\mu\nu} = \text{tr}(\hat{\rho} \hat{T}^{\mu\nu})_{\text{ren}} = T_{\text{LE}}^{\mu\nu}(x) + \delta T^{\mu\nu}(\partial\beta, \partial\xi). \quad (29)$$

Hereby we focus our attention on the zero-order term, which can be rewritten as

$$T_{\text{LE}}^{\mu\nu}(x) = \text{tr}(\hat{\rho}_{\text{LE}} \hat{T}^{\mu\nu}(x))_{\text{ren}} = \frac{1}{Z_{\text{LE}}} \text{tr} \left(\exp \left[- \int d\Sigma_\mu \hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu \right] \hat{T}^{\mu\nu}(x) \right)_{\text{ren}}. \quad (30)$$

As β is a function of x , the above trace cannot be calculated straightforwardly. However, in the exponent of $\hat{\rho}_{\text{LE}}$, one can make a Taylor expansion of β and ξ about the same point x where the stress-energy tensor is to be evaluated. The idea is that, at the local thermodynamical equilibrium, only the nearby points will contribute to its mean value, especially if the gradients are small. In other words, the β field is mostly uniform in the region where the stress-energy tensor correlation function is significant. Hence

$$\begin{aligned} & \exp \left[- \int d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu \right) \right] \\ & \simeq \exp \left[-\beta_\nu(x) \int d\Sigma_\mu \hat{T}^{\mu\nu} + \xi(x) \int d\Sigma_\mu \hat{j}^\mu - \frac{\partial\beta_\nu}{\partial\sigma_i}(x) \int d\Sigma_\mu \hat{T}^{\mu\nu}(\sigma_i - \sigma_{0i}) + \frac{\partial\xi}{\partial\sigma_i}(x) \int d\Sigma_\mu \hat{j}^\mu(\sigma_i - \sigma_{0i}) + \dots \right] \\ & = \exp \left[-\beta_\nu(x) \hat{P}^\nu + \xi(x) \hat{Q} - \frac{\partial\beta_\nu}{\partial\sigma_i}(x) \int d\Sigma_\mu \hat{T}^{\mu\nu}(\sigma_i - \sigma_{0i}) + \frac{\partial\xi}{\partial\sigma_i}(x) \int d\Sigma_\mu \hat{j}^\mu(\sigma_i - \sigma_{0i}) + \dots \right], \end{aligned} \quad (31)$$

where σ are the curvilinear coordinates of the hypersurface Σ at the time τ' (the point x has coordinates τ' and σ_0). In the last equality we have taken into account that the integrals of the stress-energy tensor and the current over any 3D hypersurface equal the total four-momentum and charge. Now

$$\sum_{i=1}^3 \frac{\partial\beta_\nu}{\partial\sigma_i}(x)(\sigma_i - \sigma_{0i}) = \sum_{i=1}^3 \partial_\lambda \beta_\nu(x) \frac{\partial x^\lambda}{\partial\sigma_i}(x)(\sigma_i - \sigma_{0i}) = \partial_\lambda \beta_\nu(x) \sum_{i=1}^3 t_i^\lambda(x)(\sigma_i - \sigma_{0i}),$$

where t_μ^i are the vectors tangent to the hypersurface Σ . If the β field is vorticity-free, one can choose the β frame with $n = \hat{\beta}$, thus the vectors t^i will be simply orthogonal to β . Hence, denoting with y the point with coordinates τ' and σ

$$\sum_{i=1}^3 t_i^\lambda(x)(\sigma_i - \sigma_{0i}) \simeq (y_\lambda - x_\lambda)_T,$$

where the subscript T stands for the transverse projection with respect to β ; introducing the definitions

$$D \equiv u^\mu \partial_\mu = \frac{1}{\sqrt{\beta^2}} \beta^\mu \partial_\mu = T \beta^\mu \partial_\mu \quad \nabla^\nu \equiv (g^{\mu\nu} - u^\mu u^\nu) \partial_\mu = (g^{\mu\nu} - T^2 \beta^\mu \beta^\nu) \partial_\mu \equiv \Delta^{\mu\nu} \partial_\mu \quad (32)$$

one can finally rewrite the eq. (31) as

$$\begin{aligned} & \exp \left[- \int d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu \right) \right] \\ & \simeq \exp \left[-\beta_\nu(x) \hat{P}^\nu + \xi(x) \hat{Q} - \nabla_\lambda \beta_\nu(x) \int_{T\Sigma} d\Sigma_\mu(y) \hat{T}^{\mu\nu}(y) (y^\lambda - x^\lambda) + \nabla_\lambda \xi(x) \int_{T\Sigma} d\Sigma_\mu(y) \hat{j}^\mu(y) (y^\lambda - x^\lambda) + \dots \right] \\ & = \exp \left[-\beta_\nu(x) \hat{P}^\nu + \xi(x) \hat{Q} - \frac{1}{2} (\nabla_\lambda \beta_\nu(x) - \nabla_\nu \beta_\lambda(x)) \hat{J}_x^{\lambda\nu} + \frac{1}{2} (\nabla_\lambda \beta_\nu(x) + \nabla_\nu \beta_\lambda(x)) \hat{L}_x^{\lambda\nu} + \nabla_\lambda \xi(x) \hat{d}_x^\lambda + \dots \right], \quad (33) \end{aligned}$$

where the integration - to a good approximation - can be carried out on the hyperplane $T\Sigma$ tangent to Σ at the point x ; \hat{J}_x is the angular momentum operator around the point x and

$$\begin{aligned} \hat{L}_x^{\nu\rho} & \equiv \frac{1}{2} \int_{T\Sigma} d\Sigma(y) n_\mu (y^\rho - x^\rho) \hat{T}^{\mu\nu}(y) + (\rho \leftrightarrow \nu) \\ \hat{d}_x^\lambda & \equiv \int_{T\Sigma} d\Sigma(y) n_\mu (y^\rho - x^\rho) \hat{j}^\mu(y). \end{aligned} \quad (34)$$

Although the coefficients of \hat{L} and \hat{D} vanish at global thermodynamical equilibrium, the last expression has not the correct limit at global equilibrium. The reason of this drawback is the lack of non-transverse derivatives of β multiplying the angular momentum operator. How to circumvent this problem is discussed in detail in the Appendix B, where it will be shown - for a general β field - that the correct leading order term in the expansion is indeed

$$\exp \left[-\beta_\nu(x) \hat{P}^\nu + \xi(x) \hat{Q} - \frac{1}{2} \varpi_{\lambda\nu}(x) \hat{J}_x^{\lambda\nu} + \frac{1}{2} (\nabla_\lambda \beta_\nu(x) + \nabla_\nu \beta_\lambda(x)) \hat{L}_x^{\lambda\nu} + \nabla_\lambda \xi(x) \hat{d}_x^\lambda \right], \quad (35)$$

where ϖ is given by the eq. (21).

The expression (35), once (21) is taken into account, implies that $\hat{\rho}_{\text{LE}}$ can be expanded in the gradients of the β and ξ fields with the Kubo identity, starting from a point of global thermodynamical equilibrium with constant inverse four-temperature $\beta(x)$ and chemical potential $\xi(x)T(x)$, where

$$\hat{\rho}_{\text{eq}} = \frac{1}{Z_{\text{eq}}} \text{tr}(\exp \left[-\beta_\nu(x) \hat{P}^\nu + \xi(x) \hat{Q} \right]).$$

Therefore

$$T_{\text{LE}}^{\mu\nu}(x) \simeq \frac{1}{Z_{\text{eq}}(\beta(x), \xi(x))} \text{tr} \left(\exp \left[-\beta_\nu(x) \hat{P}^\nu + \xi(x) \hat{Q} \right] \hat{T}^{\mu\nu}(x) \right)_{\text{ren}} + \mathcal{O}(\partial\beta, \partial\xi).$$

The first term of the expansion can be readily identified: it is the mean value of the stress-energy tensor at the global thermodynamical equilibrium with a *global* inverse temperature four-vector and chemical potential equal to those in x . In other words, it is the *ideal* part of the stress-energy tensor and the above expansion can be written as

$$T_{\text{LE}}^{\mu\nu}(x) \simeq T_{\text{id}}^{\mu\nu}(x) + \mathcal{O}(\nabla\beta, \nabla\xi) = (\rho + p)_{\text{eq}} \frac{1}{\beta^2} \beta^\mu(x) \beta^\nu(x) - p g^{\mu\nu} + \mathcal{O}(\partial\beta, \partial\xi), \quad (36)$$

where the energy density ρ and pressure p are the same thermodynamical functions of $\beta^2(x), \xi(x)$ as at equilibrium. The eq. (36) shows that the mean value of the stress-energy tensor differs from the ideal one by terms which, potentially, are of the first order in the gradients of β and ξ . It is presently unknown to us if they are non-vanishing. Should this be the case, one could even expect corrections to the formulae of transport coefficients.

VI. THE β FRAME VS LANDAU FRAME

We now come to a major point, namely the discussion of the difference between the β frame and the familiar Landau frame. In the previous section we have seen that, in the β frame, eq. (29) holds, and an equivalent one also holds for j :

$$T^{\mu\nu} = T_{\text{LE}}^{\mu\nu} + \delta T^{\mu\nu} \quad j^\mu = j_{\text{LE}}^\mu + \delta j^\mu$$

If the β field is non-vorticious, then, because of the eq. (11)

$$\beta_\mu \delta T^{\mu\nu} = 0 \quad \beta_\mu \delta j^\mu = 0. \quad (37)$$

Indeed, the first of the two equations (37) apparently imposes the orthogonality between the viscous part of the stress-energy tensor and the velocity vector, a condition often referred to as “Landau matching condition”, so naively one would say that the β frame and the Landau frame are equivalent, at least as long as β is vorticity-free. However, the actual definition of the Landau frame prescribes that the velocity four-vector u_L is the timelike eigenvector of T

$$T^{\mu\nu} u_{L\nu} = \lambda u_L^\mu. \quad (38)$$

It is worth remarking that the above Landau frame definition provides 4 independent equations, whereas the definition of the β frame involve 5 equations. In fact, the Landau frame definition is usually, often tacitly, supplemented by the equality of the proper charge density respectively with its local equilibrium value

$$u_L \cdot j = u_L \cdot j_{LE},$$

which indeed amounts to enforce the second equality in the eq. (37). In the traditional Landau scheme, this equation is sometimes justified through a redefinition of the temperature and chemical potential [28] in a non-equilibrium situation. However, as we have emphasized in this work, temperature and chemical potential are not ambiguous quantities in LTE. In fact, when changing frames, it should always be checked whether the basic relations involving thermodynamical quantities hold with the accordingly defined temperature and chemical potential.

The equation (37) implies the eq. (38) only if β is an eigenvector of $T_{LE}^{\mu\nu}$

$$0 = \beta_\mu \delta T^{\mu\nu} = \beta_\mu (T^{\mu\nu} - T_{LE}^{\mu\nu}) = \beta_\mu T^{\mu\nu} - \lambda \beta^\nu,$$

whence β is the timelike eigenvector of T , so $\hat{\beta} \equiv u_L$.

So, *the β frame coincides with the Landau frame if β is vorticity free and it is the timelike eigenvector of T_{LE} .* In all other cases, including the case of a vorticious β field, the Landau and β frame are not equivalent.

The β is an eigenvector of T_{LE} if $T_{LE} = T_{id}$. However, we have seen in sect. V that this is not generally the case for the quantum form of LTE, i.e. there may be corrections to the ideal stress-energy tensor depending on the gradients of the β field itself. In the Landau frame definition the starting point is the decomposition of the actual tensor T along u_L

$$T^{\mu\nu} = (\rho_L + p) u_L^\mu u_L^\nu - p g^{\mu\nu} + \Pi^{\mu\nu}, \quad (39)$$

with the understood assumption is that $\Pi \rightarrow 0$ at LTE [29] and, *a fortiori*, at the global thermodynamical equilibrium. This is not always the case, in fact there is a remarkable instance of violation of the above limit and of inequivalence between Landau and β frame: the rotational ensemble, which is a global equilibrium case. Its density operator reads (for vanishing chemical potential) [25]

$$\hat{\rho} = \frac{1}{Z} \exp[-\hat{H}/T + \omega \cdot \hat{J}_z/T] P_V,$$

where \hat{J}_z is the angular momentum operator along some fixed axis z and ω is the angular velocity (see fig. 2); P_V is a projector onto localized states, those obtained by enforcing peculiar boundary conditions on the quantum fields at some radius R of an indefinitely long cylinder with axis z and such that $\omega R < c$. This density operator has a full cylindrical symmetry along z axis and it has been extensively studied in ref. [25]. It can be put [23] in the same general form seen in eq. (5)

$$\hat{\rho}_{LE} = \frac{1}{Z} \exp \left[- \int_{\Sigma} d\Sigma_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu(x) \right) \right] \quad (40)$$

with β proportional to the rigid four-velocity field

$$\beta = \frac{1}{T} (1, \boldsymbol{\omega} \times \mathbf{x})$$

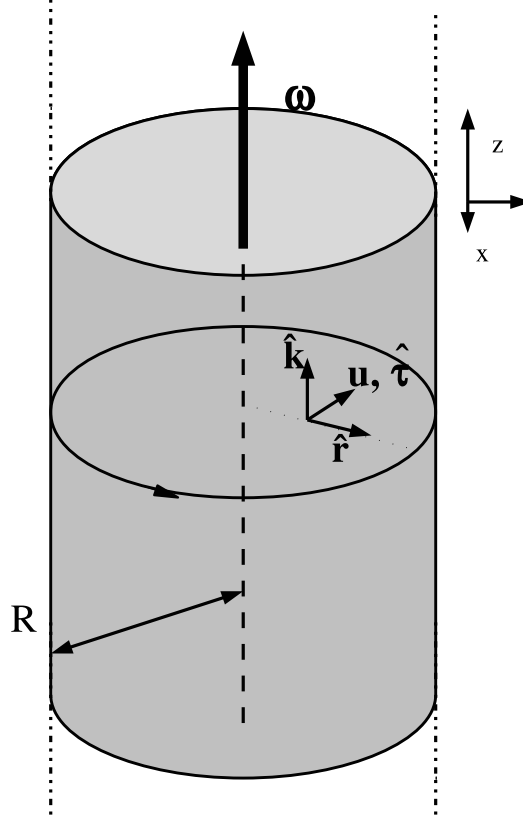


FIG. 2. Rotating cylinder with finite radius R at temperature T . Also shown the inertial frame axes and the spatial parts of the vectors of tetrad.

fulfilling the eq. (7) and whose field lines are circles centered on the z axis (see fig. 2). The density operator in (40) is therefore independent of the spacelike hypersurface Σ (i.e. time-independent) provided that the flux of $\hat{T}^{\mu\nu}\beta_\nu$ and \hat{j}^μ vanish at the boundary

$$\int_{\text{Boundary}} d\Sigma_\mu \left(\hat{T}^{\mu\nu}\beta_\nu - \xi \hat{j}^\mu \right) = 0. \quad (41)$$

Because of the cylindrical symmetry, a symmetric tensor does not need to be of the ideal form with $u = \hat{\beta}$. In fact, its most general form reads:

$$\begin{aligned} T^{\mu\nu} = & G(r)u^\mu u^\nu + H(r)(\hat{\tau}^\mu u^\nu + \hat{\tau}^\nu u^\mu) + I(r)(\hat{r}^\mu u^\nu + \hat{r}^\nu u^\mu) \\ & + J(r)\hat{\tau}^\mu \hat{\tau}^\nu + K(r)(\hat{r}^\mu \hat{\tau}^\nu + \hat{r}^\nu \hat{\tau}^\mu) + L(r)\hat{r}^\mu \hat{r}^\nu - M(r)g^{\mu\nu}, \end{aligned} \quad (42)$$

where G, H, I, J, K, L, M are generic function of the radial coordinate r such that $H(0) = I(0) = K(0) = 0$, $u = \hat{\beta}$, $\hat{r} = (0, \hat{\mathbf{r}})$, $\hat{k} = (0, \hat{\mathbf{k}})$ and $\hat{\tau}$ is the spacelike versor orthogonal to the previous three, that is

$$\hat{\tau} = (\gamma v, \gamma \hat{\mathbf{v}})$$

being $\mathbf{v} = \boldsymbol{\omega} \times \mathbf{x}$ and $\gamma = (1 - v^2)^{-1/2}$ (see fig. 2). Clearly, if one of the scalar functions H, I, J, K, L in eq. (42) is non-vanishing, the four-vector $u = \hat{\beta}$ is not the eigenvector of T and the Landau and β frame differ. Furthermore, if the scalar functions in (42) do not meet specific relations, the diagonal form of the tensor at equilibrium is not the ideal one (no isotropy) and the understood assumption $\Pi = 0$ at equilibrium for the Landau frame decomposition (39) breaks down.

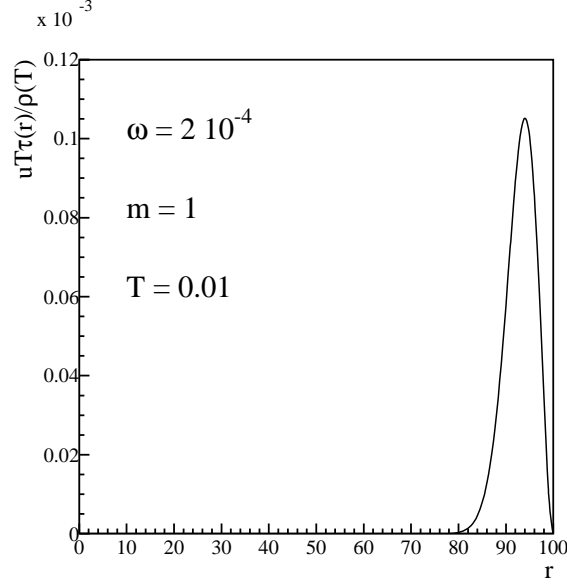


FIG. 3. Ratio between the projection $u_\mu \hat{\tau}_\nu T^{\mu\nu}$ for the free scalar field at global thermodynamical equilibrium within a rotating cylinder and the energy density of an ideal gas as a function of the radial distance r . The radius R is in arbitrary units and the values of ω , T , R and m lie in the non-relativistic domain. The ratio vanishes for $r = R$ owing to the enforced boundary condition.

As an example we calculate the stress-energy tensor of the free scalar real field (for details see Appendix A). The boundary condition at the outer surface $r = R$ of the cylinder is $\hat{\psi}(R) = 0$ which ensures the necessary vanishing of the flux eq. (41). Indeed, since $\hat{\psi}(R) = 0$, the gradient of the field at $r = R$ is normal to the outer surface, that is:

$$\partial_\mu \hat{\psi}|_{r=R} = \hat{\chi}(t, R, \phi, z) \hat{r}^\mu. \quad (43)$$

Since for the free scalar field ³

$$\hat{T}^{\mu\nu} = \partial^{(\mu} \hat{\psi} \partial^{\nu)} \hat{\psi} - g^{\mu\nu} \hat{\mathcal{L}} \quad \hat{\mathcal{L}} = \frac{1}{2} \left(\partial_\mu \hat{\psi} \partial^\mu \hat{\psi} - m^2 \hat{\psi}^2 \right) \quad (44)$$

one has

$$\hat{r}_\mu \hat{T}^{\mu\nu}(R) \beta_\nu = 0$$

hence the eq. (41), taking into account that $\xi = 0$. Furthermore, the condition (43) make, at the operatorial level, the fluxes of the energy and angular momentum outside the cylinder boundary vanishing, which amounts to have conserved \hat{H} and \hat{J}_z , as it ought to be.

Unlike for the ideal case, $T^{\mu\nu} \beta_\nu$ is not parallel to β^μ . Indeed, one has

$$\hat{\tau}_\mu T^{\mu\nu} \beta_\nu = \sqrt{\beta^2} 2\gamma^2 \sum_{M=-\infty}^{+\infty} \sum_{p_T} \int dp_L \frac{J_M^2(p_T r)}{(2\pi)^2 \varepsilon R^2 J_M'^2(p_T R)} \frac{1}{e^{\beta(\varepsilon - M\omega)} - 1} \left[\omega r \left(\varepsilon^2 + \frac{M^2}{r^2} \right) - (1 + \omega^2 r^2) \frac{\varepsilon M}{r} \right]$$

where p_T are the discrete values related to the zeroes $\zeta_{l,M}$ of the Bessel function J_M by $p_T R = \zeta_{l,M}$ and $\varepsilon = \sqrt{p_T^2 + p_L^2 + m^2}$, see Appendix A. As expected, the above expression is vanishing for $\omega = 0$, i.e. in the non-rotating case, but for $\omega \neq 0$ is non vanishing, what is confirmed by numerical computation shown in fig. 3. Indeed, its ratio to the energy density peaks at a value of order $(\omega/T)^2$ at large r .

³ The round brackets on indices stand for symmetrization.

Furthermore, the full stress-energy tensor turns out to be

$$T = \begin{pmatrix} u \cdot T \cdot u & u \cdot T \cdot \hat{\tau} & 0 & 0 \\ u \cdot T \cdot \hat{\tau} & \hat{\tau} \cdot T \cdot \hat{\tau} & 0 & 0 \\ 0 & 0 & \hat{r} \cdot T \cdot \hat{r} & 0 \\ 0 & 0 & 0 & k \cdot T \cdot k \end{pmatrix}$$

where the quoted components are non-vanishing and calculated in Appendix A. While it is possible - of course - to express the Landau timelike eigenvector u_L of T as superposition of u and τ , the two spacelike eigenvectors n and k have different eigenvalues, because (see again Appendix A):

$$\hat{r} \cdot T \cdot \hat{r} - k \cdot T \cdot k = \sum_{M=-\infty}^{+\infty} \sum_{p_T} \int dp_L \frac{2}{(2\pi)^2 \varepsilon R^2 J_M'^2(p_T R)} \frac{1}{e^{\beta(\varepsilon - M\omega)} - 1} [p_T^2 J_M'(p_T r)^2 - p_L^2 J_M(p_T r)^2],$$

which is not vanishing. This can be readily seen by setting $r = R$ and using the boundary condition of the Bessel function. Consequently, at thermodynamical equilibrium, the term Π in the decomposition (39) is non-vanishing, unlike commonly assumed; the Landau and the β frame are not equivalent in this case.

VII. EQUATIONS OF RELATIVISTIC HYDRODYNAMICS IN THE β FRAME

The β frame is an especially suitable framework to write the equations of relativistic hydrodynamics. As it is well known, the general problem is to determine the evolution of the stress-energy tensor, and possibly several vector currents starting from definite initial conditions, under the assumption of approximate local thermodynamical equilibrium. This condition, in case of one conserved current, as we have seen, reduces the number of unknown functions to 5, that is the four components of β and ξ , which is just the number of continuity equations. In terms of these variables, the equations of relativistic hydrodynamics do not show any distinction between equations of motion and equation of state (which is encoded in the dependence of the pressure on β^2 and ξ , as we will see).

A. Ideal hydrodynamics

As we have seen in sect. V the stress-energy tensor in x at the lowest order in the gradient expansion can be approximated by the ideal one $T_{\text{id}}^{\mu\nu}$ with inverse temperature four-vector and the chemical potential equal to those in the point x . We have shown in ref. [23] that it can be obtained by taking derivatives of the thermodynamic potential current $\phi^\mu = p\beta^\mu$, where p is the equilibrium pressure, a scalar field depending on the scalars β^2 and ξ . Thus

$$T_{\text{id}}^{\mu\nu} = -2 \frac{\partial p}{\partial \beta^2} \beta^\mu \beta^\nu - p g^{\mu\nu},$$

being the derivative of the pressure proportional to the proper enthalpy density

$$-2 \frac{\partial p}{\partial \beta^2} = \frac{\rho + p}{\beta^2} = \frac{h}{\beta^2}. \quad (45)$$

Similarly

$$j_{\text{eq}}^\mu = \frac{\partial p}{\partial \xi} \beta^\mu,$$

being the derivative of the pressure with respect to ξ proportional to the charge density

$$\frac{\partial p}{\partial \xi} = \frac{n}{\sqrt{\beta^2}}.$$

These expressions allow to reformulate ideal relativistic hydrodynamics through 5 unknown functions: the four-vector β (whose modulus is the inverse local temperature) and the scalar field ξ , corresponding to the ratio μ/T . There are indeed 5 differential equations corresponding to the conservation equations of T and j , which in principle allow to solve the problem, provided that the functional relation $p(\beta^2, \xi)$ is known, which is but the *complete* equation of

state. Nothing new, however the introduction of these variables as primary fields allows to gain further insight into the structure and features of relativistic hydrodynamics.

At the lowest order in the gradients, using eq. (29), the continuity equations are those of the ideal hydrodynamics

$$\partial_\mu T_{\text{id}}^{\mu\nu} = -2 \frac{\partial^2 p}{\partial \beta^{22}} \beta^\mu \beta^\nu \partial_\mu \beta^2 - 2 \frac{\partial^2 p}{\partial \beta^2 \partial \xi} \beta^\nu \beta^\mu \partial_\mu \xi - 2 \frac{\partial p}{\partial \beta^2} (\beta^\nu \partial \cdot \beta + \beta^\mu \partial_\mu \beta^\nu) - \frac{\partial p}{\partial \beta^2} \partial^\nu \beta^2 - \frac{\partial p}{\partial \xi} \partial^\nu \xi = \mathcal{O}(\partial^2) \simeq 0 \quad (46)$$

and

$$\partial_\mu j_{\text{eq}}^\mu = \frac{\partial^2 p}{\partial \xi^2} \beta^\mu \partial_\mu \xi + \frac{\partial^2 p}{\partial \beta^2 \partial \xi} \beta^\mu \partial_\mu \beta^2 + \frac{\partial p}{\partial \xi} \partial \cdot \beta = \mathcal{O}(\partial^2) \simeq 0; \quad (47)$$

Being:

$$\partial \cdot \beta = \frac{D\beta^2}{2\sqrt{\beta^2}} + \nabla \cdot \beta$$

the eq. (47) can be written, at the lowest order, as

$$\frac{\partial^2 p}{\partial \xi^2} \sqrt{\beta^2} D\xi + \frac{\partial^2 p}{\partial \beta^2 \partial \xi} \sqrt{\beta^2} D\beta^2 + \frac{\partial p}{\partial \xi} \partial \cdot \beta = 0 \quad (48)$$

and the eq. (46), at the lowest order, can be split into two equations projecting along β and transversely to it using (32)

$$\begin{aligned} \frac{\partial^2 p}{\partial \beta^{22}} \beta^2 \sqrt{\beta^2} D\beta^2 + \frac{\partial^2 p}{\partial \beta^2 \partial \xi} \beta^2 \sqrt{\beta^2} D\xi + \frac{\partial p}{\partial \beta^2} \beta^2 \left(\frac{3}{2\sqrt{\beta^2}} D\beta^2 + \nabla \cdot \beta \right) + \frac{1}{2} \frac{\partial p}{\partial \xi} \sqrt{\beta^2} D\xi &= 0 \\ \frac{\partial p}{\partial \beta^2} \left(\sqrt{\beta^2} \Delta_{\mu\nu} D\beta^\nu + \frac{1}{2} \nabla_\mu \beta^2 \right) + \frac{1}{2} \frac{\partial p}{\partial \xi} \nabla_\mu \xi &= 0. \end{aligned} \quad (49)$$

These two equations are the relativistic generalizations of the continuity equations and the Euler equation of motion of the fluid. We can readily retrieve its familiar form by noting that

$$\frac{1}{2} \frac{\partial p}{\partial \beta^2} \nabla_\mu \beta^2 + \frac{1}{2} \frac{\partial p}{\partial \xi} \nabla_\mu \xi = \frac{1}{2} \nabla_\mu p$$

and

$$\sqrt{\beta^2} \Delta_{\mu\nu} D\beta^\nu = \sqrt{\beta^2} \Delta_{\mu\nu} D \left(\frac{1}{T} u^\nu \right) = \Delta_{\mu\nu} \beta^2 D u^\nu = \beta^2 A_\mu,$$

$A^\nu = D u^\nu$ being the acceleration by definition. We then get, by using (45)

$$\frac{\partial p}{\partial \beta^2} \beta^2 A_\mu = -\frac{1}{2} (\rho + p) A_\mu = -\frac{1}{2} \nabla_\mu p$$

that is the well known form of the relativistic Euler equation.

It is interesting to note that the first term in the relativistic Euler equation in (49) can also be written as

$$\sqrt{\beta^2} \Delta_{\mu\nu} D\beta^\nu + \frac{1}{2} \nabla_\mu \beta^2 = \beta_\lambda \Delta_{\mu\nu} (\partial^\lambda \beta^\nu + \partial^\nu \beta^\lambda) \quad (50)$$

as well as

$$\sqrt{\beta^2} \Delta_{\mu\nu} D\beta^\nu + \frac{1}{2} \nabla_\mu \beta^2 = \beta^2 A^\mu + \frac{1}{2} \nabla_\mu \frac{1}{T^2} = \frac{1}{T^2} \left(A_\mu - \frac{1}{T} \nabla_\mu T \right). \quad (51)$$

One can recognize in this expression the four-vector which the heat flow q^μ is proportional to in the first order dissipative hydrodynamics. Hence, we can say that the ideal relativistic Euler equation amount to state that the first-order dissipative heat flow is parallel to the first-order dissipative current proportional to $\nabla \xi$. For an uncharged fluid, it simply states that at the first-order in the gradient expansion, this dissipative current vanishes.

We can use (48) to obtain $D\xi$ as a function of $D\beta^2$ and plug into the first equation of (49), which then becomes

$$\begin{aligned} & \left[\frac{\partial}{\partial \beta^2} ((\beta^2)^{3/2} \frac{\partial p}{\partial \beta^2}) - \frac{(\frac{\partial^2 p}{\partial \beta^2 \partial \xi})^2 (\beta^2)^{3/2}}{\frac{\partial^2 p}{\partial \xi^2}} - \frac{\frac{\partial p}{\partial \xi} \frac{\partial^2 p}{\partial \beta^2 \partial \xi}}{\frac{\partial^2 p}{\partial \xi^2}} \sqrt{\beta^2} - \frac{(\frac{\partial p}{\partial \xi})^2}{4\sqrt{\beta^2} \frac{\partial^2 p}{\partial \xi^2}} \right] D\beta^2 \\ & + \left[\frac{\partial p}{\partial \beta^2} \beta^2 - \frac{\frac{\partial p}{\partial \xi} \left(\beta^2 \frac{\partial^2 p}{\partial \beta^2 \partial \xi} + \frac{1}{2} \frac{\partial p}{\partial \xi} \right)}{\frac{\partial^2 p}{\partial \xi^2}} \right] \nabla \cdot \beta = 0. \end{aligned} \quad (52)$$

This formula allows to obtain the derivative of β^2 along the flow as a function of $\nabla \cdot \beta = (1/T)\nabla \cdot u$. Similarly, one can obtain the transverse gradient of ξ as a function of the derivatives of β through the (49). It should be kept in mind that these relations hold up to terms of the second order in the gradients. They can be used to eliminate some of the gradients in the first-order expansion of the stress-energy tensor, or, better, to replace some of the first-order gradients with transverse gradients of the β field plus further corrections of the second order.

B. Dissipative hydrodynamics in the β frame

Relativistic dissipative hydrodynamics has been the subject of intense investigations over the past decade [30–39] and an exhaustive discussion is well beyond the scope of this work. Herein, we confine ourselves to show that the β frame is best suited to approach dissipative relativistic hydrodynamics as a gradient expansion. The main reason thereof has been mentioned in sect. V, that is the eqs. (28) expressing the density operator as a function of the present time local equilibrium operator and an integral of the gradients of the β and ξ fields. The expansion has been briefly outlined in sect. V for the stress-energy tensor, but it can be extended to any observable.

We are now going to show that indeed, in the familiar 1st order (Navier-Stokes) dissipative hydrodynamics, the transverse gradients of the velocity field and of the temperature can be re-expressed in terms of the gradients of β . We have already shown (see eqs. (50,51) that the combination

$$\left(A_\mu - \frac{1}{T} \nabla_\mu T \right)$$

appearing in the familiar form of the heat current vector can be rewritten in a compact way in terms of the gradients of β . Similarly, the transverse gradients of the velocity field $\nabla^\mu u^\nu$ can be written as follows

$$\begin{aligned} \nabla_\mu u^\nu &= \nabla_\mu \frac{\beta^\nu}{\sqrt{\beta^2}} = \beta^\nu \left(-\frac{1}{2} \right) (\beta^2)^{-3/2} \nabla_\mu \beta^2 + \frac{1}{\sqrt{\beta^2}} \nabla_\mu \beta^\nu \\ &= \frac{1}{\sqrt{\beta^2}} \left(-\frac{\beta^\nu \beta^\rho}{\beta^2} \nabla_\mu \beta_\rho + \nabla_\mu \beta^\nu \right) = \frac{1}{\sqrt{\beta^2}} \Delta^{\rho\nu} \nabla_\mu \beta_\rho, \end{aligned} \quad (53)$$

where we have used the definition (32). Hence, the Navier-Stokes shear term can be fully expressed in terms of the inverse temperature four-vector. Likewise, it is easy to show that the expansion term

$$\nabla_\mu u^\mu = \frac{1}{\sqrt{\beta^2}} \nabla_\mu \beta^\mu.$$

VIII. CONCLUSIONS

We conclude with a short recapitulation of the main findings of this work:

- The notion of relativistic local thermodynamical equilibrium (LTE) can be defined independently of kinetic theory, in a form which is suitable for a strongly interacting fluid.
- Local thermodynamical equilibrium notion is, by construction, frame dependent. There is a preferred frame for it, the one where basic thermodynamics relations take on the simplest form, what we have called the β frame; the β frame is *the* frame when expansions from LTE are to be carried out.
- Physically, the β four-vector direction is identified by the four-velocity of an idealized relativistic thermometer at equilibrium with the system.

- The β frame has many interesting features in relativistic hydrodynamics, both ideal and dissipative. The β four-vector and the other intensive parameter $\xi = \mu/T$ are the solutions of the eqs. (11) for a non-vorticious β field, or, in general, of the eqs. (22).
- The β frame in general differ from both Eckart and Landau frames. It coincides with the Landau frame if the mean LTE value of the stress-energy tensor is the ideal one.

Furthermore, we have seen that the familiar ideal hydrodynamic equations of motion can be written in a form where β and ξ are the 5 unknown fields. Also, first order dissipative hydrodynamics can be written in a form where the gradients are, again, only those of β and ξ . It would be very interesting to extend the Israel-Stewart theory of causal hydrodynamics in terms of these fields and assess the stability of the equations.

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REFERENCES

- [1] R. Baier, P. Romatschke, D. T. Son, A. O. Starinets and M. A. Stephanov, JHEP **0804**, 100 (2008).
- [2] P. Kovtun, D. T. Son and A. O. Starinets, Phys. Rev. Lett. **94**, 111601 (2005).
- [3] P. F. Kolb, U. W. Heinz, P. Huovinen, K. J. Eskola and K. Tuominen, Nucl. Phys. A **696**, 197 (2001).
- [4] J. -Y. Ollitrault, Eur. J. Phys. **29**, 275 (2008).
- [5] U. Heinz and R. Snellings, Ann. Rev. Nucl. Part. Sci. **63**, 123 (2013).
- [6] R. Baier, P. Romatschke and U. A. Wiedemann, Phys. Rev. C **73**, 064903 (2006).
- [7] P. Huovinen and P. V. Ruuskanen, Ann. Rev. Nucl. Part. Sci. **56**, 163 (2006).
- [8] L. P. Csernai, J. .I. Kapusta and L. D. McLerran, Phys. Rev. Lett. **97**, 152303 (2006)
- [9] W. Florkowski, *Phenomenology of Ultra-Relativistic heavy ion collisions*, World Scientific (2010).
- [10] P. Bozek and W. Broniowski, arXiv:1403.6042 [nucl-th].
- [11] K. Damle and S. Sachdev, Phys. Rev. B **56**, 8714 (1997).
- [12] S.A. Hartnoll, P.K. Kovtun, M. Müller and S. Sachdev, Phys. Rev. B **76**, 144502 (2007)
- [13] P. Kovtun, C. Herzog, S. Sachdev and D.T. Son, Phys. Rev. D **75**, 085020 (2007)
- [14] L. Fritz, J. Schmalian, M. Muller and S. Sachdev, Phys. Rev. B **78**, 085416, (2008)
- [15] A. Adams, L. D. Carr, T. Schäfer, P. Steinberg and J. E. Thomas, New J. Phys. **14**, 115009 (2012).
- [16] J. Casalderrey-Solana, H. Liu, D. Mateos, K. Rajagopal and U. A. Wiedemann, arXiv:1101.0618 [hep-th].
- [17] G. S. Denicol, E. Molnr, H. Niemi and D. H. Rischke, Eur. Phys. J. A **48**, 170 (2012).
- [18] S. R. De Groot, W. A. van Leeuwen, Ch. G. van Weert, *Relativistic kinetic theory*, North Holland (1980).
- [19] R. Balian, *From microphysics to macrophysics*, Springer, Heidelberg (2007).
- [20] P. Van and T. S. Biro, arXiv:1305.3190 [gr-qc].
- [21] T. Chrobok and H. H. von Borzeszkowski, Gen. Rel. Grav. **38** (2006) 397;
- [22] H. A. Weldon, Phys. Rev. D **26**, 1394 (1982).
- [23] F. Becattini, Phys. Rev. Lett. **108**, 244502 (2012).
- [24] W. Israel, Annals Phys. **100**, 310 (1976); W. Israel and J. M. Stewart, Annals Phys. **118**, 341 (1979).
- [25] F. Becattini and L. Tinti, Phys. Rev. D **84**, 025013 (2011).
- [26] D. N. Zubarev, Sov. Phys. Doklady **10**, 850 (1966); D. N. Zubarev and M. V. Tokarchuk, Theor. Math. Phys. **88**, 876 (1992) [Teor. Mat. Fiz. **88N2**, 286 (1991)].
- [27] A. Hosoya, M. Sakagami and M. Takao, Annals Phys. **154**, 229 (1984).
- [28] P. Kovtun, J. Phys. A **45**, 473001 (2012).
- [29] A. Muronga, Phys. Rev. Lett. **88**, 062302 (2002) [Erratum-ibid. **89**, 159901 (2002)].
- [30] P. Huovinen and D. Molnar, Phys. Rev. C **79**, 014906 (2009).
- [31] U. W. Heinz, H. Song and A. K. Chaudhuri, Phys. Rev. C **73**, 034904 (2006).
- [32] A. Muronga, Heavy Ion Phys. **15**, 337 (2002).
- [33] A. Muronga, Phys. Rev. C **69**, 034903 (2004).
- [34] G. S. Denicol, T. Kodama, T. Koide and P. .Mota, J. Phys. G **35**, 115102 (2008);
- [35] P. Romatschke, Int. J. Mod. Phys. E **19**, 1 (2010).
- [36] P. Romatschke, Class. Quant. Grav. **27**, 025006 (2010).
- [37] K. Jensen, M. Kaminski, P. Kovtun, R. Meyer, A. Ritz and A. Yarom, Phys. Rev. Lett. **109**, 101601 (2012).
- [38] P. Van and T. S. Biro, Phys. Lett. B **709**, 106 (2012).
- [39] J. Bhattacharya, S. Bhattacharyya and M. Rangamani, JHEP **1302** (2013) 153.

APPENDIX A - STRESS-ENERGY TENSOR FOR THE FREE SCALAR FIELD

The Klein-Gordon equation of the real scalar field in cylindrical coordinates with Dirichlet boundary conditions $\widehat{\psi}(R) = 0$ has the eigenfunctions:

$$f_{\mathbf{n}} = C_{\mathbf{n}} J_M(p_T r) \exp \left[-i \left(\varepsilon_{\mathbf{n}} t - p_L z - M \phi \right) \right], \quad (54)$$

where p_L is a continuous longitudinal momentum, M is the integer angular momentum quantum number and the (discrete) transverse momenta $p_T(M, l)$ with $l = 0, 1, \dots$ are the solutions of the boundary condition equation:

$$J_M(p_T R) = 0 \quad (55)$$

In the above two equations, J_M is the Bessel function of integer order M . The $\mathbf{n} = (p_L, M, l(M))$ is the vector of quantum numbers and the energy $\varepsilon_{\mathbf{n}}$ and the normalization coefficient $C_{\mathbf{n}}$ read:

$$\varepsilon_{\mathbf{n}} = \sqrt{m^2 + p_z^2 + p_T^2} \quad C_{\mathbf{n}}^2 = \frac{1}{(2\pi)^2 \varepsilon_{\mathbf{n}} R^2 J_M'^2(p_T R)}. \quad (56)$$

The eigenfunctions $f_{\mathbf{n}}$ are orthogonal:

$$\begin{aligned} \int d^3x f_{\mathbf{n}}^* f_{\mathbf{n}'} &= C_{\mathbf{n}} C_{\mathbf{n}'} \int d^3x J_M(p_T r) J_{M'}(p'_T r) \exp \left\{ i \left[\left(\varepsilon_{\mathbf{n}} - \varepsilon_{\mathbf{n}'} \right) t - (p_L - p'_L) z - (M - M') \phi \right] \right\} \\ &= C_{\mathbf{n}} C_{\mathbf{n}'} (2\pi)^2 \delta(p_L - p'_L) \delta_{M, M'} \int_0^R dr r J_M(p_T r) J_{M'}(p'_T r) \exp \left[i \left(\varepsilon_{\mathbf{n}} - \varepsilon_{\mathbf{n}'} \right) t \right] = C_{\mathbf{n}}^2 (2\pi)^2 \frac{R^2}{2} J_M'(p_T R)^2 \delta_{\mathbf{n}, \mathbf{n}'}, \end{aligned}$$

where in the last equality we took advantage of the orthogonality relations of Bessel functions and:

$$\delta_{\mathbf{n}, \mathbf{n}'} \equiv \delta(p_L - p'_L) \delta_{M, M'} \delta_{l, l'} \quad (57)$$

The full orthogonality relations can be rewritten in the more compact form by using the normalization in eq. (56):

$$\int d^3x f_{\mathbf{n}}^* f_{\mathbf{n}'} = \frac{1}{2\varepsilon_{\mathbf{n}}} \delta_{\mathbf{n}, \mathbf{n}'}. \quad (58)$$

Another useful relation is:

$$\begin{aligned} \int d^3x f_{\mathbf{n}}^* f_{\mathbf{n}'}^* &= C_{\mathbf{n}} C_{\mathbf{n}'} \int d^3x J_M(p_T r) J_{M'}(p'_T r) \exp \left\{ i \left[\left(\varepsilon_{\mathbf{n}} + \varepsilon_{\mathbf{n}'} \right) t - (p_L + p'_L) z - (M + M') \phi \right] \right\} \\ &= C_{\mathbf{n}} C_{\mathbf{n}'} (2\pi)^2 \delta(p_L + p'_L) \delta_{M, -M'} \int_0^R dr r J_M(p_T r) J_{M'}(p'_T r) \exp \left[i \left(\varepsilon_{\mathbf{n}} + \varepsilon_{\mathbf{n}'} \right) t \right] \\ &= \frac{1}{2\varepsilon_{\mathbf{n}}} (-1)^M \exp \left(2i \varepsilon_{\mathbf{n}} t \right) \delta_{\mathbf{n}', \tilde{\mathbf{n}}}, \end{aligned} \quad (59)$$

where $\tilde{\mathbf{n}} = (-p_L, -M, l)^4$ and in the last equality we used the $J_{-M} = (-1)^M J_M$ relation among the integer Bessel funtions.

The field operator reads:

$$\widehat{\psi}(x) = \sum_{\mathbf{n}} \left[f_{\mathbf{n}} a_{\mathbf{n}} + f_{\mathbf{n}}^* a_{\mathbf{n}}^\dagger \right]. \quad (60)$$

From eqs. (58), (59) and the canonical equal time commutation relations:

$$[\widehat{\psi}(t, \mathbf{x}), \widehat{\psi}(t, \mathbf{y})] = [\partial_t \widehat{\psi}(t, \mathbf{x}), \widehat{\Pi}(t, \mathbf{y})] = 0 \quad [\widehat{\psi}(t, \mathbf{x}), \widehat{\Pi}(t, \mathbf{y})] = i \delta^3(\mathbf{x} - \mathbf{y}), \quad (61)$$

the commutation relations between creation and annihilation operators $a_{\mathbf{n}}^\dagger, a_{\mathbf{n}}$ follow:

$$[a_{\mathbf{n}}, a_{\mathbf{n}'}] = 0 \quad [a_{\mathbf{n}}^\dagger, a_{\mathbf{n}'}^\dagger] = 0 \quad [a_{\mathbf{n}}, a_{\mathbf{n}'}^\dagger] = \delta_{\mathbf{n}, \mathbf{n}'} \quad (62)$$

⁴ It is important to note how $\varepsilon_{\tilde{\mathbf{n}}} = \varepsilon_{\mathbf{n}}$ and $C_{\tilde{\mathbf{n}}} = C_{\mathbf{n}}$

Reasoning as in ref. [25], one can readily show that:

$$\langle a_{\mathbf{n}} a_{\mathbf{n}'} \rangle = \langle a_{\mathbf{n}}^\dagger a_{\mathbf{n}'}^\dagger \rangle = 0 \quad \langle a_{\mathbf{n}}^\dagger a_{\mathbf{n}'} \rangle = \frac{1}{e^{\beta(\varepsilon_{\mathbf{n}} - M\omega)} - 1} \delta_{\mathbf{n}, \mathbf{n}'}, \quad (63)$$

where in the last term one can recognize the typical Bose statistics mean occupation number henceforth denoted as n_B :

$$n_B \equiv \frac{1}{e^{\beta(\varepsilon_{\mathbf{n}} - M\omega)} - 1}$$

We can now calculate the projections of the mean stress-energy tensor $\langle : \hat{T} : \rangle$ with \hat{T} like in eq. (44), in the base $\{u, n, k, \tau\}$. First, we calculate the mean value $\langle : \hat{\mathcal{L}} : \rangle$ of the Lagrangian density in ref. (44); for this purpose, one needs derivatives of the field:

$$\begin{aligned} \partial_r \hat{\psi} &= \sum_{\mathbf{n}} \left[(\partial_r f_{\mathbf{n}}) a_{\mathbf{n}} + (\partial_r f_{\mathbf{n}}^*) a_{\mathbf{n}}^\dagger \right] & \partial_z \hat{\psi} &= \sum_{\mathbf{n}} (ip_L) \left[f_{\mathbf{n}} a_{\mathbf{n}} - f_{\mathbf{n}}^* a_{\mathbf{n}}^\dagger \right] \\ \partial_\varphi \hat{\psi} &= \sum_{\mathbf{n}} (iM) \left[f_{\mathbf{n}} a_{\mathbf{n}} - f_{\mathbf{n}}^* a_{\mathbf{n}}^\dagger \right]. \end{aligned}$$

Dialing the above expansions in the lagrangian in eq. (44), one obtains:

$$\begin{aligned} \langle : \hat{\mathcal{L}} : \rangle &= \frac{1}{2} \langle : (\partial_t \hat{\psi})^2 - (\partial_x \hat{\psi})^2 - (\partial_y \hat{\psi})^2 - (\partial_z \hat{\psi})^2 - m^2 \hat{\psi}^2 : \rangle = \frac{1}{2} \langle : (\partial_t \hat{\psi})^2 - (\partial_r \hat{\psi})^2 - \frac{1}{r^2} (\partial_\varphi \hat{\psi})^2 - (\partial_z \hat{\psi})^2 - m^2 \hat{\psi}^2 : \rangle \\ &= \sum_{\mathbf{n}} n_B \left\{ \varepsilon_{\mathbf{n}}^2 |f_{\mathbf{n}}|^2 - |\partial_r f_{\mathbf{n}}|^2 - \frac{M^2}{r^2} |f_{\mathbf{n}}|^2 - p_L^2 |f_{\mathbf{n}}|^2 - m^2 |f_{\mathbf{n}}|^2 \right\} = \sum_{\mathbf{n}} n_B \left\{ \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 - |\partial_r f_{\mathbf{n}}|^2 \right\}. \end{aligned}$$

Every projection involving one k is vanishing as $k \cdot \partial = \partial_z$ involves a multiplication of each term within the sum $\sum_{\mathbf{n}}$ by p_L . On the other hand, the $k \cdot T \cdot k$ diagonal term reads:

$$k \cdot T \cdot k = \langle : (\partial_z \hat{\psi})^2 + \mathcal{L} : \rangle = \sum_{\mathbf{n}} n_B \left\{ 2 p_L^2 |f_{\mathbf{n}}|^2 + \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 - |\partial_r f_{\mathbf{n}}|^2 \right\}.$$

Similarly, for the projections along n , the off-diagonal $\hat{r} \cdot T \cdot \hat{\tau}$ and $\hat{r} \cdot T \cdot u$ terms vanish because:

$$\begin{aligned} \langle : \partial_{(t} \hat{\psi} \partial_{r)} \hat{\psi} : \rangle &= \sum_{\mathbf{n}} n_B \left\{ -i \varepsilon_{\mathbf{n}} f_{\mathbf{n}}^* (\partial_r f_{\mathbf{n}}) + i \varepsilon_{\mathbf{n}} f_{\mathbf{n}}^* (\partial_r f_{\mathbf{n}}) \right\} = 0 \\ \langle : \partial_{(\varphi} \hat{\psi} \partial_{r)} \hat{\psi} : \rangle &= \sum_{\mathbf{n}} n_B \left\{ i M f_{\mathbf{n}} (\partial_r f_{\mathbf{n}}^*) - i M f_{\mathbf{n}}^* (\partial_r f_{\mathbf{n}}) \right\} = 0, \end{aligned}$$

taking into account that $f_{\mathbf{n}} (\partial_r f_{\mathbf{n}}^*)$ is real. On the other hand:

$$\hat{r} \cdot T \cdot \hat{r} = \langle : (\partial_r \hat{\psi})^2 + \hat{\mathcal{L}} : \rangle = \sum_{\mathbf{n}} n_B \left[|\partial_r f_{\mathbf{n}}|^2 + \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 \right].$$

Using the relations:

$$u \cdot \partial = \gamma \partial_t + \gamma v \frac{1}{r} \partial_\varphi = \gamma \partial_t + \gamma \omega \partial_\varphi \quad \hat{\tau} \cdot \partial = \gamma v \partial_t + \gamma \frac{1}{r} \partial_\varphi,$$

we can calculate the diagonal projections onto u and $\hat{\tau}$:

$$\begin{aligned}
u \cdot T \cdot u &= \varepsilon = \langle : \gamma^2 \left[(\partial_t \hat{\psi})^2 + \omega^2 (\partial_\varphi \hat{\psi})^2 + 2\omega (\partial_t \hat{\psi} \partial_\varphi \hat{\psi}) \right] - \mathcal{L} : \rangle = \\
&= \sum_{\mathbf{n}} n_B \left\{ 2\gamma^2 \left[\varepsilon_{\mathbf{n}}^2 + M^2 \omega^2 - 2\varepsilon_{\mathbf{n}} M \omega \right] |f_{\mathbf{n}}|^2 - \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 + |\partial_r f_{\mathbf{n}}|^2 \right\} = \\
&= \sum_{\mathbf{n}} n_B \left\{ 2\gamma^2 \left[\varepsilon_{\mathbf{n}} - M \omega \right]^2 |f_{\mathbf{n}}|^2 - \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 + |\partial_r f_{\mathbf{n}}|^2 \right\}
\end{aligned} \tag{64}$$

$$\begin{aligned}
\hat{\tau} \cdot T \cdot \hat{\tau} &= \langle : \gamma^2 \left[v^2 (\partial_t \hat{\psi})^2 + \left(\frac{1}{r} \partial_\varphi \hat{\psi} \right)^2 + 2\omega (\partial_t \hat{\psi} \partial_\varphi \hat{\psi}) \right] + \mathcal{L} : \rangle = \\
&= \sum_{\mathbf{n}} n_B \left\{ 2\gamma^2 \left[\varepsilon_{\mathbf{n}}^2 \omega^2 r^2 + \frac{M^2}{r^2} - 2\varepsilon_{\mathbf{n}} M \omega \right] |f_{\mathbf{n}}|^2 + \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 - |\partial_r f_{\mathbf{n}}|^2 \right\} = \\
&= \sum_{\mathbf{n}} n_B \left\{ 2\gamma^2 \left[\varepsilon_{\mathbf{n}} v - \frac{M}{r} \right]^2 |f_{\mathbf{n}}|^2 + \left(p_T^2 - \frac{M^2}{r^2} \right) |f_{\mathbf{n}}|^2 - |\partial_r f_{\mathbf{n}}|^2 \right\}.
\end{aligned} \tag{65}$$

as well as the non-diagonal term:

$$\begin{aligned}
u \cdot T \cdot \hat{\tau} &= \langle : \gamma^2 \left\{ v \left[\left(\partial_t \hat{\psi} \right)^2 + \frac{1}{r^2} \left(\partial_\varphi \hat{\psi} \right)^2 \right] + (1 + v^2) \partial_t \hat{\psi} \partial_\varphi \hat{\psi} \right\} : \rangle = \\
&= 2\gamma^2 \sum_{\mathbf{n}} n_B \left\{ \omega r \left(\varepsilon_{\mathbf{n}}^2 + \frac{M^2}{r^2} \right) - (1 + \omega^2 r^2) \frac{\varepsilon_{\mathbf{n}} M}{r} \right\} |f_{\mathbf{n}}|^2,
\end{aligned}$$

or, writing explicitly the $|f_{\mathbf{n}}|^2$ function

$$u \cdot T \cdot \hat{\tau} = 2\gamma^2 \sum_{\mathbf{n}} \frac{J_M^2(p_T r)}{(2\pi)^2 \varepsilon_{\mathbf{n}} R^2 J_M'^2(p_T R)} \frac{1}{e^{\beta(\varepsilon_{\mathbf{n}} - M\omega)} - 1} \left[\omega r \left(\varepsilon_{\mathbf{n}}^2 + \frac{M^2}{r^2} \right) - (1 + \omega^2 r^2) \frac{\varepsilon_{\mathbf{n}} M}{r} \right]. \tag{66}$$

APPENDIX B - CORRECT LOCAL EQUILIBRIUM EXPANSION

We want to expand the exponent in the formula (5) for the case of general β field, i.e. not restricted to the condition of non-vorticity (see sect. III):

$$- \int d\Sigma n_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu \right)$$

For our purposes, we can set $\xi = 0$. The general result with $\xi \neq 0$ shall be obtained by adding the corresponding terms in eq. (35) to our final expression.

If $n \neq \hat{\beta}$, the tangent vectors to the Σ hypersurface are not orthogonal to β ; in fact, they are orthogonal to the vector b , as we have seen in sect. III, which is related to β with the eq. (20). It is thus convenient to use the decomposition in eq. (20) to rewrite the above integral as:

$$\begin{aligned}
- \int d\Sigma n_\mu \left(\hat{T}^{\mu\nu} \beta_\nu - \xi \hat{j}^\mu \right) &= - \int d\Sigma(y) n_\mu \left(\hat{T}^{\mu\nu} (b_\nu + \varpi_{\nu\lambda} y^\lambda) - \xi \hat{j}^\mu \right) \\
&= - \int d\Sigma(y) n_\mu \left(\hat{T}^{\mu\nu} b_\nu - \frac{1}{2} \varpi_{\lambda\nu} (y^\lambda \hat{T}^{\mu\nu} - y^\nu \hat{T}^{\mu\lambda}) - \xi \hat{j}^\mu \right)
\end{aligned} \tag{67}$$

We can now carry out a Taylor expansion of the integrand starting from the point x up to the first order derivatives in b ; in view of (21), the zero-order approximation for $\varpi(y)$ is sufficient. Hence, eq. (67) gives rise to:

$$\begin{aligned}
&- \int d\Sigma(y) n_\mu \left(\hat{T}^{\mu\nu} b_\nu - \frac{1}{2} \varpi_{\lambda\nu} (y^\lambda \hat{T}^{\mu\nu} - y^\nu \hat{T}^{\mu\lambda}) - \xi \hat{j}^\mu \right) \\
&\simeq -b_\nu(x) \int d\Sigma(y) n_\mu \hat{T}^{\mu\nu} - \frac{\partial b_\nu}{\partial x^\rho} \int d\Sigma(y) n_\mu (y^\rho - x^\rho)_T \hat{T}^{\mu\nu} - \frac{1}{2} \varpi_{\lambda\nu}(x) \int d\Sigma(y) n_\mu (y^\lambda \hat{T}^{\mu\nu} - y^\nu \hat{T}^{\mu\lambda}) \\
&= -b_\nu(x) \hat{P}^\nu - \frac{\partial b_\nu}{\partial x^\rho} \int d\Sigma(y) n_\mu (y^\rho - x^\rho)_T \hat{T}^{\mu\nu} + \frac{1}{2} \varpi_{\lambda\nu}(x) \hat{J}^{\lambda\nu}
\end{aligned} \tag{68}$$

where we have used the orthogonality between the tangent vectors to Σ and b implied by the equality in eq. (20). As the b field has vanishing external derivative (see eq. (19)), only the symmetric part of the last integral can be retained in the last expression of (68):

$$-b_\nu(x)\hat{P}^\nu + \frac{1}{2}\varpi_{\lambda\nu}(x)\hat{J}^{\lambda\nu} - \frac{1}{2}(\nabla_\lambda\beta_\nu + \nabla_\nu\beta_\lambda)\hat{L}_x^{\lambda\nu} \quad (69)$$

where \hat{L} is defined in eq. (34).

By using the identity:

$$\hat{J}^{\lambda\nu} = \hat{J}_x^{\lambda\nu} + x^\lambda\hat{P}^\nu - x^\nu\hat{P}^\lambda$$

where \hat{J}_x stands for the total angular momentum around the point x (instead of the origin), and by using the relation (20), one can rewrite eq. (69) as:

$$-\beta_\nu(x)\hat{P}^\nu + \frac{1}{2}\varpi_{\lambda\nu}(x)\hat{J}_x^{\lambda\nu} - \frac{1}{2}(\nabla_\lambda\beta_\nu + \nabla_\nu\beta_\lambda)\hat{L}_x^{\lambda\nu} \quad (70)$$

It can be seen that this expression has the correct global equilibrium case. Indeed, at global equilibrium, the coefficient of \hat{L} vanish as β is a Killing vector field according to (7) and $\varpi = \text{const}$ is given by the external derivative of the β field like in the eq. (21). By reintroducing ξ , we obtain the correct formula for the LTE exponential (35).